APPROXIMATE EVALUATION OF AN EXPRESSION ARISING IN THE THEORY OF TIME-DELAY ESTIMATION

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SUMMARY

In a previous paper (Ref. 1) a formula was derived for the greatest lower bound of the variance of unbiased estimates of the time delay between transmission and reception of a waveform, when the received waveform is observed in a background of additive white Gaussian noise.

The present paper evaluates this expression approximately for a class of wave forms.

In Ref. 1, the following problem (among others) was discussed: let $F(t)$ be a real-valued function defined over $-\infty < t < \infty$; let the 'received waveform' be

$$v(t) = \alpha_o F(t - \tau) + n(t) \tag{1}$$

where α_o is a known real positive number; τ is an unknown real number belonging to a certain a-priori interval $[a,b]$; and $n(t)$ is white Gaussian noise with spectral density N_o.*

Let $\sigma^2_{glb}(\tau_o)$ denote the greatest lower bound for the variance of unbiased estimates of τ, when the true value is τ_o. An expression for σ^2_{glb} was derived in Ref. 1 (under certain conditions); the result was as follows:

Let

$$R = \frac{2\alpha_o^2}{N_o} \int_{-\infty}^{\infty} F^2(t)\, dt \tag{2}$$

$$\rho(\tau) = \frac{\int_{-\infty}^{\infty} F(t)\, F(t+\tau)\, dt}{\int_{-\infty}^{\infty} F^2(t)\, dt} \tag{3}$$

$$L(\tau) = \exp\left[R\, \rho(\tau)\right] - 1 \tag{4}$$

(where exp( ) is the exponential function)

* See Ref. 1 for a more precise mathematical formulation.

$$\mathcal{L}(u) = \int_{-\infty}^{\infty} e^{-iu\tau} L(\tau)\, d\tau \tag{5}$$

Then for sufficiently large R, and for $(\tau_0 - a)$ and $(b - \tau_0)$ sufficiently large,

$$\sigma^2_{glb}(\tau_0) \approx \frac{e^{-R}}{2\pi} \int_{-\infty}^{\infty} \frac{\mathcal{L}'^2(u)}{\mathcal{L}(u)}\, du \tag{6}$$

The purpose of the present paper is to derive approximate expressions for the integral occurring on the right side of Eq. 6, for a large class of functions $\rho(\tau)$. The results will correspond closely with ones intuitive expectations.

The following cases will be treated:

<u>Case A</u>

$$\rho(\tau) = \bar{\rho}(\tau) \cos \omega_0 \tau \tag{7}$$

with

$$\bar{\rho}(\tau) \approx 1 - \frac{1}{2} \beta^2 \tau^2 + \ldots \tag{8}$$

where the remainder in the expansion of $\bar{\rho}(\tau)$ is sufficiently small near $\tau = 0$.

<u>Case B</u>

$$\rho(\tau) = \bar{\rho}(\tau) \cos \omega_0 \tau \tag{9}$$

with

$$\bar{\rho}(\tau) \approx 1 - \gamma|\tau| + \ldots , \tag{10}$$

where the remainder in the expansion of $\bar{\rho}(\tau)$ is sufficiently small near $\tau = 0$.

We will first evaluate the results for $\omega_o = 0$. For Case A, $\omega_o = 0$,

$$L(\tau) \approx e^R \exp\left[-\frac{1}{2} R \beta^2 \tau^2\right] - 1 \tag{11}$$

For sufficiently large R,

$$\mathcal{L}(u) \approx e^R \sqrt{\frac{2\pi}{R\beta^2}} \exp\left[\frac{-u^2}{2R\beta^2}\right] \tag{12}$$

and

$$\frac{e^{-R}}{2\pi} \int_{-\infty}^{\infty} \frac{\mathcal{L}'^2(u)}{\mathcal{L}(u)} du \approx \frac{1}{\beta^2 R} \tag{13}$$

For Case B, $\omega_o = 0$,

$$L(\tau) \approx e^R e^{-R\gamma|\tau|} - 1 \tag{14}$$

For sufficiently large R,

$$\mathcal{L}(u) \approx \frac{2 e^R R\gamma}{u^2 + R^2\gamma^2} \tag{15}$$

and

$$\frac{e^{-R}}{2\pi} \int_{-\infty}^{\infty} \frac{\mathcal{L}'^2(u)}{\mathcal{L}(u)} du \approx \frac{1}{2 R^2\gamma^2} \tag{16}$$

For $\omega_o \neq 0$, we proceed as follows (assuming throughout that $R >> 1$):

$$\exp\left[R\,\bar{\rho}(t)\cos\omega_o t\right] = \exp\left\{\frac{R\bar{\rho}(t)}{2}\left[e^{i\omega_o t} + e^{-i\omega_o t}\right]\right\} \tag{17}$$

$$= \sum_{n=-\infty}^{\infty} I_n\left[R\bar{\rho}(t)\right] e^{in\omega_o t}$$

where I_n is the modified Bessel function of the first kind of order n.(2)

Now, let

$$I_n^*(x) = I_n(x), \qquad n \neq 0 \tag{18}$$

$$= I_o(x) - 1, \qquad n = 0$$

Then, as is well known(2),

$$I_n^*(x) = \frac{1}{\pi}\int_0^{\pi}\left\{\exp\left[x\cos\theta\right] - 1\right\}\cos n\theta\, d\theta \tag{19}$$

Also, by (17),

$$L(t) = \exp\left[R\,\bar{\rho}(t)\cos\omega_o t\right] - 1 = \sum_{-\infty}^{\infty} I_n^*\left[R\,\bar{\rho}(t)\right] e^{in\omega_o t} \tag{20}$$

Therefore

$$\mathcal{L}(u) = \int_{-\infty}^{\infty}\left\{\exp\left[R\,\bar{\rho}(t)\cos\omega_o t\right] - 1\right\} e^{-iut}\, dt \tag{21}$$

$$= \mathcal{Re}\int_{-\infty}^{\infty}\sum_{-\infty}^{\infty} I_n^*\left[R\,\bar{\rho}(t)\right] e^{it(n\omega_o - u)}\, dt$$

$$= \mathcal{R}e \int_{-\infty}^{\infty} \sum_{-\infty}^{\infty} \frac{1}{\pi} \int_{0}^{\pi} \left\{ \exp\left[R\,\bar{\rho}(t) \cos\theta \right] - 1 \right\} \cos n\theta \; e^{it(n\,\omega_o - u)} \, d\theta \, dt$$

$$= \mathcal{R}e \frac{1}{\pi} \sum_{-\infty}^{\infty} \int_{0}^{\pi} \cos n\theta \int_{-\infty}^{\infty} \left\{ \exp\left[R\,\bar{\rho}(t) \cos\theta \right] - 1 \right\} e^{it(n\,\omega_o - u)} \, dt \, d\theta$$

$$= \frac{1}{\pi} \sum_{-\infty}^{\infty} \int_{0}^{\pi} \cos n\theta \int_{-\infty}^{\infty} \left\{ \exp\left[R\,\bar{\rho}(t) \cos\theta \right] - 1 \right\} \cos\left[(n\,\omega_o - u)t \right] dt \, d\theta$$

<u>Case A: $\bar{\rho}(\tau) \approx 1 - (1/2)\beta^2\tau^2$</u>

$$\int_{-\infty}^{\infty} \left\{ \exp\left[R\,\bar{\rho}(t) \cos\theta \right] - 1 \right\} \cos\left[(n\,\omega_o - u)t \right] dt \qquad (22)$$

$$\approx 2\int_{0}^{\infty} \exp\left[R \cos\theta \left(1 - \tfrac{1}{2}\beta^2 t^2\right) \right] \cos\left[(n\,\omega_o - u)t \right] dt \qquad (\text{for } \cos\theta > 0)$$

$$\approx e^{R\cos\theta} \sqrt{\frac{2\pi}{R\,\beta^2 \cos\theta}} \; \exp\left[\frac{-(n\,\omega_o - u)^2}{2\,R\,\beta^2 \cos\theta} \right] \qquad (\text{for } \cos\theta > 0)$$

In (21), we may neglect the contribution to the integral for values of θ such that $\cos\theta$ is negative. Thus,

$$\mathcal{L}(u) \qquad (23)$$

$$\approx \left[\frac{\pi}{2} R\,\beta^2 \right]^{-\frac{1}{2}} \int_{0}^{\frac{\pi}{2}} \sum_{-\infty}^{\infty} \cos n\theta \; \exp\left[\frac{-(n\,\omega_o - u)^2}{2\,R\,\beta^2 \cos\theta} \right] \frac{e^{R\cos\theta}}{\sqrt{\cos\theta}} \, d\theta$$

Because of the $e^{R \cos \theta}$ term in the integral, we may assume that for $R >> 1$, only the portion of the integral near $\theta = o$ is significant. Using $\cos \theta \approx 1 - \frac{1}{2}\theta^2$, we arrive at

$$\mathcal{L}(u) \approx e^{R} \left[\frac{\pi}{2} R \beta^2\right]^{-\frac{1}{2}} \sum_{-\infty}^{\infty} \exp\left[\frac{-(n\,\omega_o - u)^2}{2\,R\,\beta^2}\right] \tag{24}$$

$$X \int_0^{\infty} \cos n\theta \, \exp\left\{-\theta^2\left[\frac{R}{2} + \frac{(u - n\,\omega_o)^2}{4\,R\beta^2}\right]\right\} d\theta$$

For each n, we may neglect the term $\frac{(u - n\,\omega_o)^2}{4\,R\,\beta^2}$ in the exponential function in the integral in Eq. (24); this is true because, if u is such that $\frac{(u - n\,\omega_o)^2}{4\,R\,\beta^2} \gtrsim \frac{R}{2}$, then $\frac{(u - n\,\omega_o)^2}{2\,R\,\beta^2} \gtrsim R$, so that the exponential outside of the integral in Eq. (24) is $\lesssim e^{-R}$.

Thus, carrying out the integration,

$$\mathcal{L}(u) \approx \frac{e^R}{R\beta}\sum_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\,R\beta^2}\left[n^2(\omega_o^2 + \beta^2) - 2un\,\omega_o + u^2\right]\right\} \tag{25}$$

$$\approx \sum_{-\infty}^{\infty} \frac{e^R}{R\beta} \exp\left\{-\frac{\omega_o^2 + \beta^2}{2\,R\beta^2}\left[n - \frac{u\,\omega_o}{\omega_o^2 + \beta^2}\right]^2\right\} \exp\left[\frac{-u^2}{2R(\omega_o^2 + \beta^2)}\right]$$

We will now also assume that $\omega_o >> \beta$; then

$$\mathcal{L}(u) \approx \frac{e^R}{R\beta} \exp\left[\frac{-u^2}{2R\,\omega_o^2}\right] \sum_{-\infty}^{\infty} \exp\left[\frac{-\omega_o^2}{2R\beta^2}\left(n - \frac{u}{\omega_o}\right)^2\right] \tag{26}$$

We must now distinguish two regions,

<u>Region a</u>: $\dfrac{\omega_o^2}{2R\,\beta^2} >> 1$

<u>Region b</u>: $\dfrac{\omega_o^2}{2R\,\beta^2} << 1$

(Always subject to $R >> 1$.)

In Region a, $\mathcal{L}(u)$ can be written as

$$\mathcal{L}(u) = \frac{e^R}{R\beta} \exp\left[\frac{-u^2}{2R\,\omega_o^2}\right] \sum_{-\infty}^{\infty} \mathcal{L}_n(u) \tag{27}$$

where

$$\mathcal{L}_n(u) = \exp\left[\frac{-\omega_o^2}{2R\,\beta^2}\left(n - \frac{u}{\omega_o}\right)^2\right] \tag{28}$$

and, approximately,

$$\mathcal{L}_n(u)\ \mathcal{L}_m(u) \approx 0 \quad \text{for} \quad m \neq n \tag{29}$$

Also

$$\frac{\mathcal{L}'^2(u)}{\mathcal{L}(u)} \approx \frac{e^R}{R^3\beta} \sum_{-\infty}^{\infty} \exp\left[\frac{-u^2}{2R\,\omega_o^2}\right] \exp\left[\frac{-(u - n\,\omega_o)^2}{2R\,\beta^2}\right] \times \left[\frac{(u - n\,\omega_o)^2}{\beta^4} + \frac{2u(u - n\,\omega_o)}{\beta^2\,\omega_o^2} + \frac{u^2}{\omega_o^4}\right] \tag{30}$$

Now consider the integral from $-\infty$ to ∞ of the n^{th} term in the sum in Eq. (30):

$$\int_{-\infty}^{\infty} \exp\left[\frac{-u^2}{2R\,\omega_o^2}\right] \exp\left[\frac{-(u - n\,\omega_o)^2}{2R\,\beta^2}\right] \times \left[\frac{(u - n\,\omega_o)^2}{\beta^4} + \frac{2u(u - n\,\omega_o)}{\beta^2\,\omega_o^2} + \frac{u^2}{\omega_o^4}\right] du \tag{31}$$

$$\approx \exp\left[\frac{-n^2}{2R}\right]\left\{\int_{-\infty}^{\infty}\exp\left[\frac{-(u-n\,\omega_o)^2}{2R\,\beta^2}\right]\left[\frac{(u-n\,\omega_o)^2}{\beta^4}+\frac{2n\,\omega_o}{\beta^2\,\omega_o^2}(u-n\omega_o)+\frac{n^2}{\omega_o^2}\right]du\right\}$$

$$\approx \exp\left[\frac{-n^2}{2R}\right]\left[\frac{R^{3/2}\sqrt{2\pi}}{\beta}\right]\left[1+\frac{n^2}{R^2}\;\frac{\beta^2 R}{\omega_o^2}\right]$$

Since $R >> 1$ and $\frac{\beta^2 R}{\omega_o^2} << 1$ in Region a, we can now replace the sum over n by an integral and neglect the term in $\frac{n^2}{R^2}\cdot\frac{\beta^2 R}{\omega_o^2}$, giving

$$\int_{-\infty}^{\infty}\frac{\mathcal{L}'^2(u)}{\mathcal{L}(u)}\,du \approx \left[\frac{e^R}{R^3\beta}\right]\left[\frac{R^{3/2}\sqrt{2\pi}}{\beta}\right]\int_{-\infty}^{\infty}\exp\left[\frac{-v^2}{2R}\right]dv \approx \frac{2\pi e^R}{R\,\beta^2} \tag{32}$$

So that. finally, in Region a, and assuming $\omega_o >> \beta$,

$$\sigma^2_{glb} \approx \frac{e^{-R}}{2\pi}\int_{-\infty}^{\infty}\frac{\mathcal{L}'^2(u)}{\mathcal{L}(u)}\,du \approx \frac{1}{R\beta^2} \tag{33}$$

Comparing with Eq. (13), we see that this is precisely the inherent error variance associated with the envelope $\bar{\rho}(\tau)$.

In Region b, we may evaluate $\mathcal{L}(u)$ from Eq. (26) as follows: the sum in Eq. (26) may be replaced by an integral, giving

$$\sum_{-\infty}^{\infty} \exp\left\{\frac{-\omega_o^2}{2R\,\beta^2}\;(n - \frac{u}{\omega_o})^2\right\}$$

$$\approx \int_{-\infty}^{\infty} \exp\left[\frac{-\omega_o^2\, v^2}{2R\,\beta^2}\right] dv \;\approx\; \sqrt{2\pi R}\left(\frac{\beta}{\omega_o}\right) \tag{34}$$

so that

$$\mathcal{L}(u) \;\approx\; e^R \sqrt{\frac{2\pi}{R\,\omega_o^2}}\;\exp\left[\frac{-u^2}{2R\,\omega_o^2}\right] \tag{35}$$

It is then easily determined that, in Region b,

$$\sigma^2_{glb} \;\approx\; \frac{e^{-R}}{2\pi}\int_{-\infty}^{\infty} \frac{\mathcal{L}'^2(u)}{\mathcal{L}(u)}\,du \;\approx\; \frac{1}{R\,\omega_o^2} \tag{36}$$

These results can be interpreted as follows: in Region a, the minimum error variance is that associated with the envelope $\bar{\rho}(\tau)$; in Region b, it is that associated with a sinusoidal fine structure of frequency ω_o. The transition occurs at $\frac{\omega_o^2}{2R\,\beta^2} \approx 1$; that is, when the minimum error standard deviation associated with the envelope becomes roughly equal to the wavelength of the fine structure.

<u>Case B: $\bar{\rho}(\tau) \approx 1 - \gamma|\tau|$</u>

We will assume throughout that $\omega_o >> \gamma$. Starting from Eq. (21) and following a line similar to Eq's (22) - (26), we obtain

$$\mathcal{L}(u) \approx \frac{e^R}{R\gamma}\sqrt{\frac{2}{\pi R}}\;\sum_{-\infty}^{\infty} \exp\left[\frac{-n^2}{2R}\right]\left[1 + \left(\frac{n\,\omega_o - u}{R\,\gamma}\right)^2\right]^{-1} \tag{37}$$

We must now distinguish three regions:

<u>Region a</u> $\qquad \frac{\omega_o}{R\gamma} >> 1$

Region b $\frac{1}{\sqrt{R}} << \frac{\omega_o}{R\gamma} < < 1$

Region c $\frac{1}{\sqrt{R}} >> \frac{\omega_o}{R\gamma}$

In Region a,

$$\frac{\mathcal{L}'^2(u)}{\mathcal{L}(u)} \approx \frac{4e^R}{R^5\gamma^5}\sqrt{\frac{2}{\pi R}}\sum_{-\infty}^{\infty}\frac{(n\omega_o - u)^2}{\left[1+\left(\frac{n\omega_o - u}{R\gamma}\right)^2\right]^3}\exp\left[\frac{-n^2}{2R}\right] \qquad (38)$$

$$\int_{-\infty}^{\infty}\frac{\mathcal{L}'^2(u)}{\mathcal{L}(u)}\,du \approx \frac{e^R}{2R^2\gamma^2}\sqrt{\frac{2\pi}{R}}\sum_{-\infty}^{\infty}\exp\left[\frac{-n^2}{2R}\right] \qquad (39)$$

and replacing the sum by an integral,

$$\sigma^2_{glb} \approx \frac{e^{-R}}{2\pi}\int_{-\infty}^{\infty}\frac{\mathcal{L}'^2(u)}{\mathcal{L}(u)}\,du \approx \frac{1}{2R^2\gamma^2} \qquad (40)$$

Comparing with Eq. (16), we see that this is just the minimum error variance associated with the envelope $\bar{\rho}(\tau)$.

In Region b,

$$\mathcal{L}(u) \approx \frac{e^R}{R\gamma}\sqrt{\frac{2}{\pi R}}\int_{-\infty}^{\infty}\left[1+\left(\frac{v\omega_o - u}{R\gamma}\right)^2\right]^{-1}\exp\left[\frac{-v^2}{2R}\right]dv \qquad (41)$$

$$\approx \frac{e^R}{R\gamma}\sqrt{\frac{2}{\pi R}}\exp\left[\frac{-u^2}{2R\,\omega_o^2}\right]\int_{-\infty}^{\infty}\frac{dv}{1+\left(\frac{v\omega_o - u}{R\gamma}\right)^2}$$

$$\approx e^R\sqrt{\frac{2\pi}{R\,\omega_o^2}}\exp\left[\frac{-u^2}{2R\,\omega_o^2}\right]$$

and

$$\sigma^2_{glb} \approx \frac{e^{-R}}{2\pi} \int_{-\infty}^{\infty} \frac{\mathcal{L}'^2(u)}{\mathcal{L}(u)} du \approx \frac{1}{R\,\omega_o^2} \tag{42}$$

This is just the minimum error variance associated with a sinusoidal fine structure of frequency ω_o, but neglecting the fact that when $\bar{\rho}$ is as in Case B, the fine structure of ρ has non-zero slope at the origin.

In Region c,

$$\mathcal{L}(u) \approx \frac{e^R}{R\gamma}\sqrt{\frac{2}{\pi R}} \int_{-\infty}^{\infty} \left\{1 + \left[\frac{v\omega_o - u}{R\gamma}\right]^2\right\}^{-1} \exp\left[\frac{-v^2}{2R}\right] dv \tag{43}$$

$$\approx \frac{e^R}{R\gamma}\sqrt{\frac{2}{\pi R}} \frac{1}{1+\left(\frac{u}{R\gamma}\right)^2} \int_{-\infty}^{\infty} \exp\left[\frac{-v^2}{2R}\right] dv$$

$$\approx \frac{2e^R}{R\gamma} \frac{1}{1+\left(\frac{u}{R\gamma}\right)^2}$$

and

$$\sigma^2_{glb} \approx \frac{e^{-R}}{2\pi} \int_{-\infty}^{\infty} \frac{\mathcal{L}'^2(u)}{\mathcal{L}(u)} du \approx \frac{1}{2R^2\gamma^2} \tag{44}$$

This is the same as in Region a, and reflects the fact that the slope of ρ, sufficiently near the origin, is the same as the slope of $\bar{\rho}$.

It appears that it would be possible to carry out much the same sort of analysis for $\rho(\tau)$ of the form

$$\rho(\tau) = \rho_1(\tau)\cos\omega_o\tau + \rho_2(\tau)\sin\omega_o\tau \tag{45}$$

where $\rho_1(\tau)$ and $\rho_2(\tau)$ can be expanded in a suitable manner at the origin. this would appear, however, to be much more tedious and complicated.

REFERENCES

1. Swerling, P., 'A Method of Computing the Inherent Accuracy With Which a Time Delay Can be Measured', The RAND Corporation P-1185, 27 September 1957.

2. McLachlan, W W., 'Bessel Functions for Engineers', Oxford University Press, 1934.